①

Answers to midterm exam
KWANTUM FYSICA 1   28-9-2012

T1

a) $\langle\hat{A}\rangle(t) = \langle\psi_0| e^{+\frac{i}{\hbar}\hat{H}t}\hat{A}\, e^{-\frac{i}{\hbar}\hat{H}t}|\psi_0\rangle =$

$$\left(\sqrt{\tfrac{2}{3}}\, e^{+\frac{i}{\hbar}E_1 t}\langle\varphi_1| + e^{-i\varphi}\sqrt{\tfrac{1}{3}}\, e^{+\frac{i}{\hbar}E_2 t}\langle\varphi_2|\right)\hat{A}\left(\sqrt{\tfrac{2}{3}}\, e^{-\frac{i}{\hbar}E_1 t}|\varphi_1\rangle + e^{+i\varphi}\sqrt{\tfrac{1}{3}}\, e^{-\frac{i}{\hbar}E_2 t}|\varphi_2\rangle\right)$$

$$= \tfrac{2}{3}\langle\varphi_1|\hat{A}|\varphi_1\rangle + \tfrac{1}{3}\langle\varphi_2|\hat{A}|\varphi_2\rangle + \tfrac{\sqrt{2}}{3} e^{-i\left(\left(\frac{E_2-E_1}{\hbar}\right)t-\varphi\right)}\langle\varphi_1|\hat{A}|\varphi_2\rangle + \tfrac{\sqrt{2}}{3} e^{+i\left(\left(\frac{E_2-E_1}{\hbar}\right)t-\varphi\right)}\langle\varphi_2|\hat{A}|\varphi_1\rangle$$

$$= \frac{\sqrt{2}}{3}\left(e^{+i\left(\left(\frac{E_2-E_1}{\hbar}\right)t-\varphi\right)} + e^{-i\left(\left(\frac{E_2-E_1}{\hbar}\right)t-\varphi\right)}\right) A_0$$

$$= \frac{2\sqrt{2}}{3} A_0 \cos\left(\frac{E_2-E_1}{\hbar}\cdot t - \varphi\right) \Rightarrow$$

It only oscillates at angular frequency $\frac{E_2-E_1}{\hbar}$ with amplitude $\frac{2\sqrt{2}}{3}A_0$.

b) $P_2 = |\langle\psi(t)|\varphi_2\rangle|^2 = \left|\left(\sqrt{\tfrac{2}{3}}\langle\varphi_1| e^{+\frac{i}{\hbar}E_1 t} + e^{-i\varphi}\sqrt{\tfrac{1}{3}}\langle\varphi_2| e^{+\frac{i}{\hbar}E_2 t}\right)|\varphi_2\rangle\right|^2$

use $\langle\varphi_1|\varphi_2\rangle=0$, $\langle\varphi_2|\varphi_2\rangle=1$: $= \left|\sqrt{\tfrac{1}{3}}\, e^{i\left(\frac{E_2}{\hbar}t-\varphi\right)}\right|^2 = \frac{1}{3}$.

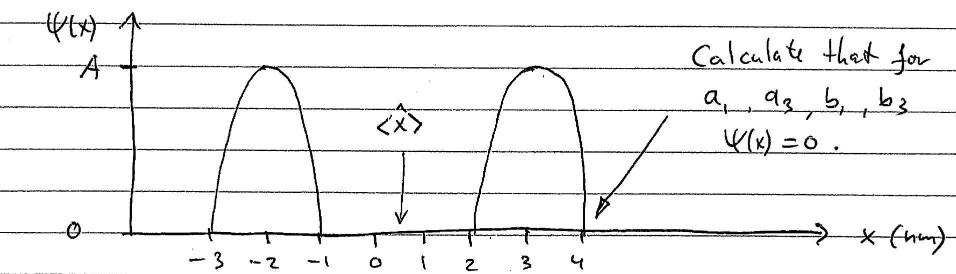
c) After the measurement the system is in state $|\varphi_2\rangle$

$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}|\varphi_2\rangle = e^{-\frac{i}{\hbar}E_2(t-t_0)}|\varphi_2\rangle$ with $t_0 = 11$ ns

This is effectively $|\psi(t)\rangle = |\varphi_2\rangle$ since the prefactor is only a global phase factor.

②

T2 Make first a sketch of the wave function



Calculate that for a_1, a_3, b_1, b_3 $\psi(x)=0$.

a) $\langle\hat{x}\rangle = \int_{-\infty}^{\infty}\psi(x)^* \times \psi(x)\,dx = \frac{1}{2}$ nm (Read from plot)

(or calculate it as

$$\langle\hat{x}\rangle = A^2\int_{a_1}^{a_3}\left(1-(x-a_2)^2\right)^2 x\,dx + A^2\int_{b_1}^{b_3}\left(1-(x-b_2)^2\right) x\,dx$$)

b) Read from the plot that $|\psi(x)|^2$ is zero for $-0.6\text{ nm} < x < 0.6$ nm $\Rightarrow$ This probability is zero

(OR calculate it as $P = \int_{-0.6\,\text{nm}}^{+0.6\,\text{nm}} |\psi(x)|^2 dx = 0$)

c) Read from the plot that $3\text{ nm} < x < 4\text{ nm}$ concerns $\frac{1}{4}$ of the probability in the plot $\Rightarrow$ the probability is $\frac{1}{4}$.

(OR calculate it as $P = \int_{3\,\text{nm}}^{4\,\text{nm}} |\psi(x)|^2 dx$

$$= \int_{b_2}^{b_3} A^2\left(1-(x-b_2)^2\right)^2 dx = \frac{1}{4}$$)

T3

3

a) $\hat{H}\,\varphi(x) = E_i\,\varphi(x) \quad \Rightarrow$

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi(x)}{dx^2} + \left(B_0\cos(3x) + k_0 x^2\right)\varphi(x) = E_i\,\varphi(x)$$

b) Use here the x-representation, for case that the single degree of freedom is the position in x direction of particle with mass m. (but you can work it out in a similar way for any other single degree of freedom)

Time-dependent Schrödinger equation: $i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \hat{H}\,\Psi(x,t)$ (1)

with for this case $\hat{H} = V(x) - \frac{\hbar^2}{2m}\frac{d^2}{dx^2}$.

Investigate whether there are solutions of the type $\Psi(x,t) = \varphi(x)\,\chi(t)$

Filling this in into Eq.(1), and deviding by $\varphi(x)\,\chi(t)$ gives

$$\frac{i\hbar}{\chi(t)}\frac{d\chi(t)}{dt} = \frac{1}{\varphi(x)}\left(-\frac{\hbar^2}{2m}\frac{d^2\varphi(x)}{dx^2}\right) + V(x)$$

This equality can only hold (left function of t only, right function of x only) if left and right are equal to a constant, which will be denoted as E_i. This gives two equations

$$\begin{cases} \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\varphi(x) = \hat{H}\varphi(x) = E_i\,\varphi(x) & \rightarrow \text{time-independent Schrödinger Eq.} \\ i\hbar\frac{d\chi(t)}{\chi(t)} = E_i\,dt \Rightarrow i\hbar\ln(\chi(t)) = E_i\cdot t + C \Rightarrow \chi(t) = e^{-\frac{i}{\hbar}(E_i t + C)} & \rightarrow \text{time evolution of states with fixed } E_i. \end{cases}$$